Verbally closed subgroups of free groups

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Abstract

We prove that every verbally closed subgroup of a free group F of a finite rank is a retract of F.

1 Introduction

Algebraically closed objects play an extremely important part in modern algebra. In this paper we study verbally closed and algebraically closed subgroups of free groups.

Recall, that if $\mathcal{K}$ is a class of structures in a language L then a structure $A \in \mathcal{K}$ is called *algebraically closed* in $\mathcal{K}$ if for any positive existential sentence $\phi(x_1, \ldots, x_n)$ in the language L with constants from A if ϕ holds in some $B \in \mathcal{K}$ that contains A then it holds in A. We refer to [17] for general facts on algebraically closed structures. An interesting particular case occurs when A is algebraically closed in $\mathcal{K} = \{A, B\}$ for some B containing A as a substructure $A \leq B$. In this event A is termed *algebraically closed* in B. Another typical and useful variation on algebraically closed structures appears when one restricts the definition above onto ϕ from a fixed subset Φ of positive existential sentences from L, in which case one gets Φ-algebraically closed structures. One the other hand, if instead of positive existential sentences one has in the definitions above arbitrary existential sentences ϕ then this defines an *existentially closed* structures in $\mathcal{K}$.

In the case of groups the notions above can be explained in pure algebraic terms. To this end we remind some terminology. For groups H and G we write $H \leq G$ if H is a subgroup of G and refer to this as an *extension* of H to G. Let $X = \{x_1, x_2, ..., x_k, ...\}$ be countable infinite set of variables, and $F(X)$ be the free group with basis X. By an equation with variables $x_1, ..., x_n \in X$ and constants from H we mean an arbitrary expression $E(x_1, ..., x_n, H) = 1$ where $E(x_1, ..., x_n, H)$ is a word in the alphabet $X^{\pm 1} \cup H$, in other words $E(x_1, ..., x_n, H)$ lies in $F[H] = F(X) * H$, the free product of $F(X)$ and H. In the case when the left side of the equation does not depend from H we will omit H in its expression. We say that $E(x_1, \ldots, x_n, H) = 1$ has a solution in G if there is a substitution $x_i \to g_i$ for some elements $g_i \in G$ such that $E(g_1, \ldots, g_n) = 1$ in G. It is easy to see that a subgroup H is algebraically closed in G if and only if for every finite system of equations $S = \{E_i(x_1, \ldots, x_n, H) = 1 \mid i = 1, \ldots, m\}$ with constants from H the following holds: if S has a solution in G then it has a solution in H. By the same token, a group H is algebraically closed in a class of groups $\mathcal{K}$ if and only if H is algebraically closed in every extension $H \leq G$ with $G \in \mathcal{K}$. Replacing systems of equations in the definitions above by

systems of equations and inequalities with coefficients in H one gets the notion of existentially closed groups in $\mathcal{K}$, as well as all corresponding variations.

Groups algebraically (existentially) closed in the class of all groups were introduced by Scott in [37]. They have been thoroughly studied in 1970-80's, see, for example, papers by Macintyre [25], Eklof and Sabbagh [13], Belegradek [5, 6], Ziegler [42]; and books by Hodges [16] and Higman and Scott [14]. Nowadays, a lot more is known about groups algebraically or existentially closed in various specific classes of groups $\mathcal{K}$, in particular, when $\mathcal{K}$ consists of various nilpotent, solvable, or locally finite groups. For details we refer to a survey by Leinen [23]. On the other hand, not much is known about algebraically or existentially closed groups in the classes of groups with some presence of negative curvature, for example, in the classes of groups universally equivalent to a given hyperbolic group. To this end we would like to mention a work by Jaligot and Ould Houcine [18] on existentially closed CSA-groups (see [29] for definitions and various properties of CSA groups). Notice, that groups universally equivalent to a given torsion-free hyperbolic group are CSA.

Our interest to this topic is twofold. The first part comes from studying Krull dimension and Cantor-Bendixon rank of groups. To explain, recall first that a subgroup H of a group G is called a *retract* of G, if there is a homomorphism (termed *retraction*) $\phi : G \to H$ which is identical on H. In Section 2, Proposition 2.2, we show (and it is easy) that every retract of G is algebraically closed in G. Furthermore, if G is finitely presented and H is finitely generated then the converse is also true. This result still holds for finitely generated G which are *equationally Noetherian* (for definition see Section 2 below). However, to characterize existentially closed subgroups one needs a stronger condition. Namely, an extension $H \leq G$ is called *discriminating* if for every finite subset $K \subseteq G$ there is a retraction $\phi : G \to H$ such that the restriction of ϕ onto K is injective. It is easy to see again that if $H \leq G$ is discriminating then the subgroup H is existentially closed in G; and furthermore, if G is finitely generated relative to H and H is equationally Noetherian then the converse is also true (Proposition 2.3). If a group G is equationally Noetherian then Zariski topology on its Cartesian product (affine space) G^k, defined by algebraic sets as a pre-basis of closed sets, is Noetherian [2]. It was shown in [31] that in this case the Zariski dimension of irreducible algebraic sets Y in G^k is equal to the Krull dimension of their coordinate groups G_Y. Here the *Krull dimension* of G_Y is defined (as usual) as the supremum of the lengths of chains $p_0 \subset p_1 \subset \ldots \subset p_k$ of distinct prime ideals p_i in G_Y, where a prime ideal in G_Y is a normal subgroup N of G_Y such that $N \cap G = 1$ (the subgroup G naturally embeds into G_Y and hence into G_Y/N) and $G \leq G_Y/N$ is a discriminating extension.

Another part of our interest in various versions of algebraic closures comes from research on verbal width (or length) of elements in groups. To explain we need some notation. For $w = w(x_1, \ldots, x_n) \in F(X)$ and a group G by $w[G]$ we denote the set of all w-elements in G, i.e., $w[G] = \{w(g_1, \ldots, g_n) \mid g_1, \ldots, g_n \in G\}$. The *verbal subgroup* $w(G)$ is the subgroup of G generated by $w[G]$. The *w-width* (or *w-length*) $l_w(g) = l_{w,G}(g)$ of an element $g \in w(G)$ is the minimal natural number n such that g is a product of n w-elements in G or their inverses; the width of $w(G)$ is the supremum of widths of its elements. Usually, it is very hard to compute the w-length of a given element $g \in w(G)$ or the width of $w(G)$. The first question of this type goes back to the Ore's paper [34] where he asked whether the commutator length (i.e., the $[x, y]$-length) of every element in

a non-abelian finite simple group is equal to 1 (*Ore Conjecture*). Only recently the conjecture was established by Liebeck, O'Brian, Shalev and Tiep [24]. For recent spectacular results on the w-length in finite simple groups, we refer to the papers [21], [36] and a book [35]. For instance, A. Shalev [36] proved that for any nontrivial word w, every element of every sufficiently large finite simple group is a product of three values of w.

Two important questions arise naturally for an extension $H \leq G$ and a given word $w \in F(X)$:

- when it is true that $w(H) = w(G) \cap H$ or $w[H] = w[G] \cap H$?
- when $l_{w,G}(h) = l_{w,H}(h)$ for a given $h \in w(H)$?

To approach these questions we introduce a new notion of *verbally closed* subgroups.

Definition 1.1. A subgroup H of G is called *verbally closed* if for any word $w \in F(X)$ and $h \in H$ an equation $w(x_1, \ldots, x_n) = h$ has a solution in G if and only if it has a solution in H, i.e., $w[H] = w[G] \cap H$ for every $w \in F(X)$.

Notice, that verbally closed subgroups fit in the general picture of algebraic closures, where the closure operator is defined by the set Φ of all single equations of the type $w(x_1, \ldots, x_n) = h$, where $w \in F(X)$ and $h \in H$. In general, single equations do not suffice to get the standard algebraic closures in groups (see examples in the class of 2-nilpotent torsion-free groups due to Baumslag and Levin [4]).

Not much is known in general about verbally closed subgroups of a given group G. For instance, the following basic questions are still open for most non-abelian groups:

- Is there an algebraic description of verbally closed subgroups of G?
- Is the class of verbally closed subgroups of G closed under intersections?
- Does there exist the verbal closure $vcl(H)$ of a given subgroup H of G? Here $vcl(H)$ is the least (relative to inclusion) verbally closed subgroup of G containing H.
- If H is a finitely generated subgroup of G is $vcl(H)$ (if it exists) also finitely generated?
- Given $H \leq G$ can one find the generators of $vcl(H)$ (if it exists) effectively?

In this papers we address all the questions above in the case of a free group G. In Section 3 we prove the main result of the paper that answers (for free groups) to the first question above:

Theorem 1. *Let F be a free group of a finite rank. Then for a subgroup H of F the following conditions are equivalent:*

a) H is a retract of F.

b) H is a verbally closed subgroup of F.

c) H is an algebraically closed subgroup of F.

This result clarifies the nature of verbally or algebraically closed subgroups in F. Surprisingly, the "weak" verbal closure operator in this case is as strong as the standard one. Since quite a lot is known about retracts of a free group one can now easily derive some corollaries of the main result. The proof of the theorem is rather short, but it is based on several deep known results. Firstly we use the fact, due to Lee, that every non-abelian free group of finite rank has C-test words [22]. Secondly, precise values of the commutator verbal length of the derived subgroups in free nilpotent groups play an important part here.

In Section 4 we study verbal (= algebraic) closures of subgroups in a given nonabelian free group F_r of rank r. It immediately follows from Theorem 1 that verbally (algebraically) closed subgroup of F_r are finitely generated. Furthermore, the intersection of an arbitrary family of verbally (algebraically) closed subgroups in F_r is again verbally (algebraically) closed (see Proposition 4.1), which proves the following theorem.

Theorem 2. *Let H be a subgroup of a free group F_r with basis $\{f_1, ..., f_r\}$. . Then there exists a unique minimal (with respect to inclusion) verbally closed subgroup $vcl(H)$ of F_r containing H. The subgroup $vcl(H)$ is also the unique minimal algebraically closed subgroup in F_r containing H.*

Observe, that free factors of F_r are, of course, retracts, but the converse is not true. Particular series of such examples (with some other interesting properties) are constructed by Martino and Ventura [28] and Ciobanu and Dicks [10].

At the end of the section we study some algorithmic questions related to verbal closures in free groups. The main results are collected in the following theorem.

Theorem 3. *Let F_r be a free group with basis $\{f_1, ..., f_r\}$. Then the following holds:*

a) There is an algorithm to decide if a given finitely generated subgroup of F_r is verbally (algebraically) closed or not.

b) There is an algorithm to construct $vcl(H)$, i.e., to find a basis of $vcl(H)$ for a given finitely generated subgroup H of F_r.

We note, in passing, that Diekert, Gutierrez, and Hagenah gave an algorithm to solve equations with rational constraints in free groups [12], so given an extension $H \leq F$, where F is a free group of a finite rank, one can check algorithmically whether or not an equation $E(x_1, \ldots, x_n, F) = 1$ (with coefficients in F) has a solution in F, provided some fixed distinguished variables satisfy an extra requirement $x_i \in H$. This gives a useful complementary tool to deal with algorithmic problems related to H.

Recently, in [18] Ould Houcine and Vallino studied another notion of an algebraic closure of a subset A of a group G, which is reminiscent to adding roots of a polynomial in one variable in a field. In this case, an element b is termed *algebraic* over A if there is a formula $\phi(x)$ of group language such that $\phi(b)$ holds in G and there are only finitely many other elements in G satisfying ϕ. The set $ac(A)$ of all algebraic over A elements forms a subgroup of G. How much this subgroup relates to the algebraic or verbal closure of A - is not clear. However, there is one common component in all the variations of algebraic closures discussed here - all of them form algebraic extensions in

the sense of [20, 32]. By definition subgroups $H \le K$ of a free group F form an *algebraic extension* if H is not a subgroup of a proper free factor of K, i.e., there is no "purely transcendenyal" non-trivial extension over H in K. Since every finitely generated subgroup of F has only finitely many such algebraic extensions and one can find all of them effectively, this gives an approach to algorithmic problems for all types of algebraic closures and extensions in free groups.

At the end of the paper (Section 5) we discuss some related open problems.

2 Preliminaries

In this section we collect some known or simple facts on verbally, algebraic or existentially closed subgroups of groups.

At the beginning we mention a few simple, but useful general results. Recall that a group G is called *equationally Noetherian* if for any n every system of equations in n variables with coefficients from G is equivalent (has the same solution set in G) to some finite subsystem of itself [2, 3].

Definition 2.1. An extension $H \le G$ is called *discriminating* if for every finite subset $K \subseteq G$ there is a retraction $\phi : G \to H$ such that the restriction of ϕ onto K is injective.

Proposition 2.2. Let $H \le G$ be a group extension. Then the following holds:

1) If H is a retract of G then H is algebraically closed in G.

2) Suppose G is finitely presented and H is finitely generated. Then H is algebraically closed in G if and only if H is a retract.

3) Suppose G and H are finitely generated and H is equationally Noetherian. Then H is algebraically closed in G if and only if H is a retract.

Proof. Let $\pi : G \to H$ be a retraction. Then if a finite system of equations $\Phi(x_1, \ldots, x_n, H)$ holds in G on elements $g_1, \ldots, g_n$ then $\Phi(x_1, \ldots, x_n, H)$ holds in H on elements $\pi(g_1), \ldots, \pi(g_n)$, which proves 1).

To prove 2) assume that H is generated by a finite set $h_1, \ldots, h_m$ and G has a finite presentation $G = \langle a_1, ..., a_n \mid r_1, \ldots, r_s \rangle$. For $i = 1, ..., m$ fix a presentation $h_i = v_i(a_1, ..., a_n)$ of h_i as a word in the generators of G. Then the system of equations

$$\begin{array}{l} v_1(x_1, ..., x_n) = h_1, \ldots, v_m(x_1, ..., x_n) = h_m, \\ r_1(x_1, ..., x_n) = 1, \ldots, r_s(x_1, ..., x_n) = 1 \end{array} \qquad (1)$$

with constants $h_i \in H$ and variables $x_1, ..., x_n$ has a solution $x_1 = a_1, \ldots, x_n = a_n$ in G. Hence it has a solution $x_1 = b_1, \ldots, x_n = b_n$ in H. Now, the map $a_1 \to b_1, \ldots a_n \to b_n$ defines a retraction of G onto H, as claimed.

3) is similar to 2) (see also the argument in the proof of 2) in Proposition 2.3).

□

Proposition 2.3. Let $H \le G$ be a group extension. Then the following holds:

1) If $H \le G$ is discriminating then the subgroup H is existentially closed in G.

2) Suppose that G is finitely generated relative to H and H is equationally Noetherian. Then H is existentially closed in G if and only if the extension $H \le G$ is discriminating.

Proof. Let $H \le G$ be a discriminating extension. Suppose some elements $a_1, \ldots, a_n \in G$ satisfy in G a given finite system $\Psi(x_1, \ldots, x_n, H)$ of equations and inequalities with constants from H. Then there is a retraction $\pi : G \to H$ such that $\pi(a_1), \ldots, \pi(a_n)$ satisfy precisely the same systems of equations and inequalities, i.e., $\Psi(x_1, \ldots, x_n, H)$ holds in H on $\pi(a_1), \ldots, \pi(a_n)$. This proves 1).

2) was proven in [2], but we give a quick sketch of the proof here. Let $B_n = \{b_1, \ldots, b_n\}$ be a finite generating set of G relative to H. Denote by $R = R(b_1, \ldots, b_n, H) = 1$ a set of defining relations of G relative to $B_n \cup H$. One may correspond to this set a system of equations $\Phi = \Phi(x_1, \ldots, x_n, H)$ in variables $x_1, \ldots, x_n$ and constants from H. Since H is equationally Noetherian the system Φ is equivalent in H to some finite subsystem, say Φ_0. A given finite system Ψ of inequalities in G can be rewritten into an equivalent finite system Ψ_0 of inequalities in $X_n \cup H$. Since H is existentially closed in G the finite system of equations and inequalities $\Phi_0 \cup \Psi_0$ has a solution in H. This solution gives a retraction $G \to H$ which discriminates a given finite set of elements in G (which was encoded in the system Ψ_0.)

□

Lemma 2.4. *All types of extensions introduced above are transitive, i.e., every chain of extensions of a given type results in an extension of the same type.*

Proof. Directly from the definitions.

□

3 Description of verbally (algebraically) closed subgroups of free groups

We start with several remarks. A subgroup R of G is a retract if and only if it has a normal complement N in G, i.e. a normal subgroup N of G such that $G = RN$ and $R \cap N = 1$. It is easy to see that every direct or free factor of G is a retract. In particular, the trivial subgroup of G is a retract.

An element a of a free abelian group A_n with basis $\{a_1, ..., a_n\}$ is called *primitive* if it can be included into some basis of A_n. It is known that $a = a_1^{k_1} ... a_n^{k_n}$, where $k_1, ..., k_n \in \mathbb{Z}$, is primitive if and only if $gcd(k_1, ..., k_n) = 1$.

Lemma 3.1. *Let F_r be a free group of rank r, and $H = gp(h)$ is a cyclic subgroup of F_r generated by a non-trivial element $h \in F_r$. Then the following conditions are equivalent:*

1) H is verbally closed in F_r;

2) H is a retract of F_r;

3) the image of h in the abelianization $F_r/[F_r, F_r]$ is primitive.

Proof. Let $\{f_1, ..., f_r\}$ be a basis of F_r. The element $h \in F_r$ can be expressed uniquely in the form

$$h = f_1^{k_1} ... f_r^{k_r} h'(f_1, \dots, f_r), \tag{2}$$

where $k_1, ..., k_r \in \mathbb{Z}$ and $h'(f_1, \dots, f_r)$ is a product of commutators of words in $f_1, \dots, f_n$.

To show that 1) $\to$ 3) assume that h has a non primitive image in $F_r/[F_r, F_r]$, i.e., either $h \in [F_r, F_r]$ or $gcd(k_1, ..., k_n) = d > 1$.

Suppose first that $h \in [F_r, F_r]$, so $k_1 = \ldots = k_r = 0$. Replacing each f_i by a new variable x_i in (2) one gets an equation $h = x_1^{k_1} ... x_r^{k_r} h'(x_1, \dots, x_r)$, with h as a constant from H, which has a solution in F_r. However, this equation does not have a solution in H, since H is abelian, so $h'(h_1, \dots, h_r) = 1$ for any $h_1, \dots h_r \in H$. This shows that H is not verbally closed in F_r - contradiction. So $h \notin [F_r, F_r]$. Then in this case $gcd(k_1, ..., k_r) = d > 1$. The equation

$$h = x_1^{k_1} ... x_r^{k_r} h'(x_1, \dots, x_r)$$

still has a solution in F_r, but for any $h_1, \dots h_r \in H$ one has

$$h_1^{k_1} ... h_r^{k_r} h'(h_1, \dots, h_r) = h_1^{k_1} ... h_n^{k_r} = h^{ds} \neq h,$$

for some $s \in \mathbb{Z}$. Hence, the equation does not have a solution in H, so H is not verbally closed - contradiction. This proves 1) $\to$ 3).

To show that 3) $\to$ 2) assume that h is primitive. Then there are integers $l_1, ..., l_r$ such that $k_1 l_1 + ... k_r l_r = 1$. Now we define a homomorphism $\varphi : F_r \to H = gp(h)$ by putting $\varphi(f_i) = h^{l_i}$ for $i = 1, ..., r$. Since H is abelian $\varphi(h') = 1$, so $\varphi(h) = h$ and φ is a retraction. Hence H is a retract, as claimed.

2) $\to$ 1) follows from Proposition 2.2 statement 1).

□

Below we denote by $N_{rc} = F_r/\gamma_{c+1} F_r$ a free nilpotent group of rank r and class c. As usual $\gamma_l G$ denote the lth member of the lower central series of a group G.

Proposition 3.2. Every verbally closed subgroup H of a free group F_r has rank at most r.

Proof. Suppose H is a verbally closed subgroup of F_r of rank $m > r$, so $H \simeq F_m$. Consider a free nilpotent group $N_{m3} \simeq H/\gamma_4 H = F_m/\gamma_4 F_m$ of rank m and class 3.

It is known (see for instance [35], Corollary 1.2.6) that every element g in the derived subgroup $[N_{rc}, N_{rc}]$ of a free nilpotent group N_{rc} can be written as a product of r commutators. More precisely, if $\{z_1, ..., z_r\}$ is a basis of N_{rc} then there are elements $g_1, \dots, g_r \in N_{rc}$ such that

$$g = [g_1, z_1] ... [g_r, z_r]. \tag{3}$$

Allambergenov and Roman'kov proved in [1] that in the case when $r \geq 2$ and $c \geq 3$ there is an element u_r in $[N_{rc}, N_{rc}]$ which is not equal to any product of $r - 1$ commutators in N_{rc}.

Now we pick an element $u_m \in [N_{m3}, N_{m3}]$ which can not be expressed as a product of $m - 1$ commutators in N_{m3}. Recall that $N_{m3} \simeq H/\gamma_4 H$. Denote by h a preimage of u_m in H, notice that $h \in [H, H]$. Since $H \leq F_r$ and

$[H,H] \leq [F_r, F_r]$ it follows from (3) that the element h can be presented in the form

$$h = [g_1, g_1'] \dots [g_r, g_r'] f', \tag{4}$$

where $g_1, g_1', ..., g_r, g_r' \in F_r$ and $f' \in \gamma_4 F_r$. Replace every element g_i, g_i', f' by the corresponding product $g_i(f_1, \dots, f_r), g_i'(f_1, \dots, f_r), f'(f_1, \dots, f_r)$ of elements from a fixed basis $\{f_1, \dots, f_r\}$ of F_r. The resulting equality

$$h = [g_1(y_1, \dots, y_r), g_1'(y_1, \dots, y_r)] \dots [g_r(y_1, \dots, y_r), g_r'(y_1, \dots, y_r)] f'(y_1, \dots, y_r),$$

viewed as a system in variables $y_1, \dots, y_r$ and a constant $h \in H$, has a solution in F_r, hence in H. It follows that in $N_{m3} \simeq H/\gamma_4 H$ the element h can be expressed as a product of r commutators. Since $r < m$ we get a contradiction with our choice of u_m and h. This proves the proposition.

□

Let $r \geq 2$. A non-empty word $w(z_1, ..., z_m)$ is called a *C-test word* in m letters for F_r if for any two tuples $(g_1, ..., g_m)$ and $(v_1, ..., v_m)$ of elements of F_r the following holds: if $w(g_1, ..., g_m) = w(v_1, ..., v_m) \neq 1$ then there is an element $s \in F_r$ such that $s^{-1} g_i s = v_i, i = 1, \dots, m$. In [19] Ivanov introduced and constructed first C-test words for F_r in m letters for any $r \geq 2$.

In [22] Lee constructed for each $r, m \geq 2$, a C-test word $w_r(z_1, ..., z_m)$ for F_r with the additional property that $w_r(g_1, ..., g_m) = 1$ if and only if the subgroup of F_r generated by $g_1, ..., g_m$ is cyclic. We will refer to such words as *Lee words* for F_r.

Theorem 3.3. *Every verbally closed subgroup H of a free group F_r is a retract in F_r.*

Proof. Let H be a verbally closed subgroup of F_r. The case $r = 1$ is taken care of in Lemma 3.1, so we assume that $r \geq 2$. By Proposition 3.2 H is finitely generated with basis, say $h_1, ..., h_m$, where $m \leq r$. For $m = 1$ the statement of the theorem follows from Lemma 3.1. For the rest of proof we assume that $m \geq 2$.

Let $\{f_1, ..., f_r\}$ be a basis of F_r. For $i = 1, ..., m$ fix a presentation $h_i = v_i(f_1, ..., f_r)$ of h_i as a word in the generators. To construct a retraction $F_r \to H$ we modify the argument in the proof of 2) in Proposition 2.2.

Let $w_m(z_1, ..., z_m)$ be a Lee word for F_r (for instance, constructed by Lee in [22]). An equation

$$w_m(v_1(x_1, ..., x_r), ..., v_m(x_1, ..., x_r)) = w_m(h_1, ..., h_m), \tag{5}$$

in variables $x_1, ..., x_n$ and constants $h_1, ..., h_m$ has a solution $x_1 = f_1, \dots, x_r = f_r$ in F_r. Since H is verbally closed there is a solution $x_i = g_i$ of (5) with $g_i \in H$ for $i = 1, \dots, r$, so

$$w_m(v_1(g_1, ..., g_r), ..., v_m(g_1, ..., g_r)) = w_m(h_1, ..., h_m). \tag{6}$$

Notice that the rank of $H = \langle h_1, ..., h_m \rangle$ is at least 2, so by Lee's theorem there is an element $u \in F_r$ such that

$$v_i(g_1, ..., g_r) = u^{-1} h_i u \tag{7}$$

for $i = 1, ..., m$. Therefore

$$w_m(u^{-1}h_1u, ..., u^{-1}h_mu) = u^{-1}w_m(h_1, ..., h_m)u = w_m(h_1, ..., h_m),$$

so u commutes with $h = w_m(h_1, ..., h_m)$. It follows that there is $f \in F_r$ such that $u = f^s, h = f^t$, for some $s, t \in \mathbb{Z}$. Since an equation $h = y^t$, where y is a variable and $h \in H$ is a constant, has a solution in F_r it follows that it has a solution in H. But extraction of roots is unique in free groups, so $f \in H$ and $u = f^s \in H$. Now, the equality (7) implies that

$$v_i(ug_1u^{-1}, ..., ug_ru^{-1}) = h_i,$$

for all $i = 1, \ldots, m$. This shows that a homomorphism from F_r to H defined by $f_i \to ug_iu^{-1}, i = 1, \ldots, m$ is a retraction of F_r onto H. This proves the theorem.

□

Proof of Theorem 1. By Proposition 2.2 every retract in F_r is algebraically closed in F_r, so a) $\Longrightarrow$ c). Implication c) $\Longrightarrow$ b) is obvious. Now Theorem 3.3 implies b) $\Longrightarrow$ a). Hence all the conditions in Theorem 1 are equivalent, as claimed.

4 Verbal closures of finitely generated subgroups of free groups

Let F_r be a free group of finite rank r. In [7] Bergman proved that the intersection of two retracts in F_r is itself a retract. From this it is not hard to derive that the intersection of an arbitrary collection of retracts in F_r is again a retract (see [39, Lemma 18] or [32, Proposition 4.1]). This together with Theorem 3.3 implies the following result.

Proposition 4.1. The intersection of an arbitrary family of verbally (algebraically) closed subgroups of F_r is verbally closed.

This proves Theorem 2. Theorem 3 follows from the propositions below.

Proposition 4.2. There is an algorithm to decide if a given finitely generated subgroup of F_r is verbally (algebraically) closed or not.

Proof. In the view of Theorem 3.3 it suffices to have an algorithm that decides if a given finitely generated subgroup H of F_r is a retract or not. Such an algorithm has been known in folklore for some time. The formal description of an algorithm is given in [32, Proposition 4.6]. For completeness we give a brief description of the algorithm here.

Suppose that F_r is a free group with basis $\{f_1, \ldots, f_r\}$ and let $h_1, \ldots, h_m$ be a basis of H. Suppose $h_i = v_i(f_1, \ldots, f_r)$ is a presentation of $h_i, i = 1, \ldots, m$, as a word in the generators. Then H is a retract of F_r if and only if there exist

$x_1, \ldots, x_r \in H$ such that the endomorphism ϕ of F_r defined by $\phi(f_i) = x_i$ maps H identically to itself. That is, if

$$h_i = v_i(x_1, \ldots, x_r) \tag{8}$$

for $i = 1, \ldots, m$. To decide if such ϕ exists or not it suffices to solve (8), viewed as a system of equations in variables $x_1, \ldots, x_n$ and constants $h_1, \ldots, h_m$, in the free group H. This is decidable by Makanin's algorithm [26]. This proves the result. □

Proposition 4.3. There is an algorithm to find a basis of $vcl(H)$ for a given finitely generated subgroup H of F_r.

Proof. By Theorem 3.3 it suffices to construct the unique minimal retract in F_r containing H. This is done in [32, Proposition 4.5].

□

5 Some open problems

Problem 5.1. What are verbally closed subgroup of a free nilpotent group of finite rank?

Problem 5.2. Prove that verbally closed subgroup of a torsion-free hyperbolic group are retracts.

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